

Complex Numbers and Functions

Natural is the most fertile source of Mathematical Discoveries

- Jean Baptiste Joseph Fourier

The Complex Number System

Definition:

A complex number z is a number of the form $z = a + ib$, where the symbol $i = \sqrt{-1}$ is called *imaginary unit* and $a, b \in \mathbb{R}$. a is called the *real part* and b the *imaginary part* of z , written

$$a = \operatorname{Re} z \quad \text{and} \quad b = \operatorname{Im} z.$$

With this notation, we have $z = \operatorname{Re} z + i \operatorname{Im} z$.

The set of all complex numbers is denoted by

$$C = \{ a + ib \mid a, b \in \mathbb{R} \}.$$

If $b = 0$, then $z = a + i0 = a$, is a real number. Also if $a = 0$, then $z = 0 + ib = ib$, is a imaginary number; in this case, z is called *pure imaginary number*.

Let $a + ib$ and $c + id$ be complex numbers, with $a, b, c, d \in \mathbb{R}$.

1. **Equality**

$$a + ib = c + id \quad \text{if and only if} \quad a = c \quad \text{and} \quad b = d.$$

Note:

In particular, we have $z = a + ib = 0$ if and only if $a = 0$ and $b = 0$.

2. **Fundamental Algebraic Properties of Complex Numbers**

(i). Addition

$$(a + ib) + (c + id) = (a + c) + i(b + d).$$

(ii). Subtraction

$$(a + ib) - (c + id) = (a - c) + i(b - d).$$

(iii). Multiplication

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc).$$

Remark

(a). By using the multiplication formula, one defines the *nonnegative integral power* of a complex number z as

$$z^1 = z, \quad z^2 = zz, \quad z^3 = z^2 z, \quad \dots, \quad z^n = z^{n-1} z.$$

Further for $z \neq 0$, we define the *zero power* of z is 1; that is, $z^0 = 1$.

(b). By definition, we have

$$i^2 = -1, \quad i^3 = -i, \quad i^4 = 1.$$

(iv). Division

If $c + id \neq 0$, then

$$\frac{a + ib}{c + id} = \left(\frac{ac + bd}{c^2 + d^2} \right) + i \left(\frac{bc - ad}{c^2 + d^2} \right).$$

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Remark

(a). Observe that if $a + ib = 1$, then we have

$$\frac{1}{c + id} = \left(\frac{c}{c^2 + d^2} \right) + i \left(\frac{-d}{c^2 + d^2} \right).$$

(b). For any nonzero complex number z , we define

$$z^{-1} = \frac{1}{z},$$

where z^{-1} is called the *reciprocal* of z .

(c). For any nonzero complex number z , we now define the *negative integral power* of a complex number z as

$$z^{-1} = \frac{1}{z}, \quad z^{-2} = z^{-1}z^{-1}, \quad z^{-3} = z^{-2}z^{-1}, \quad \dots, \quad z^{-n} = z^{-n+1}z^{-1}.$$

$$(d). \quad i^{-1} = \frac{1}{i} = -i, \quad i^{-2} = -1, \quad i^{-3} = i, \quad i^{-4} = 1.$$

3. More Properties of Addition and Multiplication

For any complex numbers z, z_1, z_2 and z_3 ,

(i). *Commutative Laws of Addition and Multiplication:*

$$z_1 + z_2 = z_2 + z_1;$$

$$z_1 z_2 = z_2 z_1.$$

(ii) *Associative Laws of Addition and Multiplication:*

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3;$$

$$z_1 (z_2 z_3) = (z_1 z_2) z_3.$$

(iii). *Distributive Law:*

$$z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3.$$

(iv). *Additive and Multiplicative identities:*

$$z + 0 = 0 + z = z;$$

$$z \cdot 1 = 1 \cdot z = z.$$

(v). $z + (-z) = (-z) + z = 0.$

Complex Conjugate and Their Properties

Definition:

Let $z = a + ib \in C$, $a, b \in R$. The *complex conjugate*, or briefly *conjugate*, of z is defined by

$$\bar{z} = a - ib.$$

For any complex numbers $z, z_1, z_2 \in C$, we have the following algebraic properties of the conjugate operation:

$$(i). \quad \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2,$$

$$(ii). \quad \overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2,$$

$$(iii). \quad \overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2,$$

- (iv). $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$, provided $z_2 \neq 0$,
- (v). $\overline{\overline{z}} = z$,
- (vi). $\overline{z^n} = (\overline{z})^n$, for all $n \in \mathbb{Z}$,
- (vii). $\overline{z} = z$ if and only if $\operatorname{Im} z = 0$,
- (viii). $\overline{z} = -z$ if and only if $\operatorname{Re} z = 0$,
- (ix). $z + \overline{z} = 2 \operatorname{Re} z$,
- (x). $z - \overline{z} = i(2 \operatorname{Im} z)$,
- (xi). $z\overline{z} = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2$.

Modulus and Their Properties

Definition:

The *modulus* or *absolute value* of a complex number $z = a + ib$, $a, b \in \mathbb{R}$ is defined as

$$|z| = \sqrt{a^2 + b^2}.$$

That is the positive square root of the sums of the squares of its real and imaginary parts.

For any complex numbers $z, z_1, z_2 \in \mathbb{C}$, we have the following algebraic properties of modulus:

- (i). $|z| \geq 0$; and $|z| = 0$ if and only if $z = 0$,
- (ii). $|z_1 z_2| = |z_1| |z_2|$,
- (iii). $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, provided $z_2 \neq 0$,
- (iv). $|\overline{z}| = |z| = |-z|$,
- (v). $|z| = \sqrt{z\overline{z}}$,
- (vi). $|z| \geq |\operatorname{Re} z| \geq \operatorname{Re} z$,
- (vii). $|z| \geq |\operatorname{Im} z| \geq \operatorname{Im} z$,
- (viii). $|z_1 + z_2| \leq |z_1| + |z_2|$, (*triangle inequality*)
- (ix). $||z_1| - |z_2|| \leq |z_1 + z_2|$.

The Geometric Representation of Complex Numbers

In analytic geometry, any complex number $z = a + ib$, $a, b \in \mathbb{R}$ can be represented by a point $z = P(a, b)$ in xy -plane or Cartesian plane. When the xy -plane is used in this way to plot or represent complex numbers, it is called the *Argand plane*¹ or the *complex plane*. Under these circumstances, the x - or horizontal axis is called the *axis of real number* or simply, *real axis* whereas the y - or vertical axis is called the *axis of imaginary numbers* or simply, *imaginary axis*.

¹ The plane is named for Jean Robert Argand, a Swiss mathematician who proposed the representation of complex numbers in 1806.

Furthermore, another possible representation of the complex number z in this plane is as a *vector* $\overline{OP}$. We display $z = a + ib$ as a directed line that begins at the origin and terminates at the point $P(a, b)$. Hence the modulus of z , that is $|z|$, is the distance of $z = P(a, b)$ from the origin. However, there are simple geometrical relationships between the vectors for $z = a + ib$, the negative of z ; $-z$ and the conjugate of z ; $\bar{z}$ in the Argand plane. The vector $-z$ is vector for z reflected through the origin, whereas $\bar{z}$ is the vector z reflected about the real axis.

The addition and subtraction of complex numbers can be interpreted as vector addition which is given by the *parallelogram law*. The '*triangle inequality*' is derivable from this geometric complex plane. The length of the vector $z_1 + z_2$ is $|z_1 + z_2|$, which must be less than or equal to the combined lengths $|z_1| + |z_2|$. Thus

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

Polar Representation of Complex Numbers

Frequently, points in the complex plane, which represent complex numbers, are defined by means of polar coordinates. The complex number $z = x + iy$ can be located as polar coordinate (r, θ) instead of its rectangular coordinates (x, y) , it follows that there is a corresponding way to write complex number in polar form.

We see that r is identical to the modulus of z ; whereas θ is the directed angle from the positive x -axis to the point P . Thus we have

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta,$$

where

$$r = |z| = \sqrt{x^2 + y^2},$$

$$\tan \theta = \frac{y}{x}.$$

We called θ the *argument* of z and write $\theta = \arg z$. The angle θ will be expressed in radians and is regarded as positive when measured in the *counterclockwise* direction and negative when measured *clockwise*. The distance r is never negative. For a point at the origin; $z = 0$, r becomes zero. Here θ is undefined since a ray like that cannot be constructed. Consequently, we now defined the *polar form* of a complex number $z = x + iy$ as

$$z = r(\cos \theta + i \sin \theta) \tag{1}$$

Clearly, an important feature of $\arg z = \theta$ is that it is *multivalued*, which means for a nonzero complex number z , it has an infinite number of distinct arguments (since $\sin(\theta + 2k\pi) = \sin \theta$, $\cos(\theta + 2k\pi) = \cos \theta$, $k \in \mathbb{Z}$). Any two distinct arguments of z differ each other by an integral multiple of 2π , thus two nonzero complex number $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$ are equal if and only if

$$r_1 = r_2 \quad \text{and} \quad \theta_1 = \theta_2 + 2k\pi,$$

where k is some integer. Consequently, in order to specify a *unique* value of $\arg z$, we

may restrict its value to some interval of length. For this, we introduce the concept of *principle value of the argument* (or *principle argument*) of a nonzero complex number z , denoted as $\text{Arg } z$, is defined to be the unique value that satisfies

$$-\pi \leq \text{Arg } z < \pi.$$

Hence, the relation between $\arg z$ and $\text{Arg } z$ is given by

$$\arg z = \text{Arg } z + 2k\pi, \quad k \in \mathbb{Z}.$$

Multiplication and Division in Polar Form

The polar description is particularly useful in the multiplication and division of complex number. Consider $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$.

1. Multiplication

Multiplying z_1 and z_2 we have

$$z_1 z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)).$$

When two nonzero complex are multiplied together, the resulting product has a modulus equal to the product of the modulus of the two factors and an argument equal to the sum of the arguments of the two factors; that is,

$$\begin{aligned} |z_1 z_2| &= r_1 r_2 = |z_1| |z_2|, \\ \arg(z_1 z_2) &= \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2). \end{aligned}$$

1. Division

Similarly, dividing z_1 by z_2 we obtain

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)).$$

The modulus of the quotient of two complex numbers is the quotient of their modulus, and the argument of the quotient is the argument of the numerator less the argument of the denominator, thus

$$\begin{aligned} \left| \frac{z_1}{z_2} \right| &= \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}, \\ \arg\left(\frac{z_1}{z_2}\right) &= \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2). \end{aligned}$$

Euler's Formula and Exponential Form of Complex Numbers

For any real θ , we could recall that we have the familiar Taylor series representation of $\sin \theta$, $\cos \theta$ and e^θ :

$$\begin{aligned} \sin \theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots, & -\infty < \theta < \infty, \\ \cos \theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots, & -\infty < \theta < \infty, \\ e^\theta &= 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \dots, & -\infty < \theta < \infty, \end{aligned}$$

Thus, it seems reasonable to *define*

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \dots$$

In fact, this series approach was adopted by *Karl Weierstrass* (1815-1897) in his development of the complex variable theory. By (2), we have

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \dots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \dots \\ &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots\right) = \cos\theta + i\sin\theta. \end{aligned}$$

Now, we obtain the very useful result known as *Euler's² formula* or *Euler's identity*

$$e^{i\theta} = \cos\theta + i\sin\theta. \quad (2)$$

Consequently, we can write the polar representation (1) more compactly in *exponential form* as

$$z = re^{i\theta}.$$

Moreover, by the Euler's formula (2) and the periodicity of the trigonometry functions, we get

$$\begin{aligned} |e^{i\theta}| &= 1 \quad \text{for all real } \theta, \\ e^{i(2k\pi)} &= 1 \quad \text{for all integer } k. \end{aligned}$$

Further, if two nonzero complex numbers $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, the multiplication and division of complex numbers z_1 and z_2 have exponential forms

$$\begin{aligned} z_1 z_2 &= r_1 r_2 e^{i(\theta_1 + \theta_2)}, \\ \frac{z_1}{z_2} &= \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} \end{aligned}$$

respectively.

de Moivre's Theorem

In the previous section we learned to multiply two number of complex quantities together by means of polar and exponential notation. Similarly, we can extend this method to obtain the multiplication of any number of complex numbers. Thus, if

$z_k = r_k e^{i\theta_k}$, $k = 1, 2, \dots, n$, for any positive integer n , we have

$$z_1 z_2 \cdots z_n = r_1 r_2 \cdots r_n (e^{i(\theta_1 + \theta_2 + \cdots + \theta_n)}).$$

In particular, if all values are identical we obtain

$$z^n = (re^{i\theta})^n = r^n e^{in\theta} \quad \text{for any positive integer } n.$$

Taking $r = 1$ in this expression, we then have

$$(e^{i\theta})^n = e^{in\theta} \quad \text{for any positive integer } n.$$

By Euler's formula (3), we obtain

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \quad (3)$$

² Leonhard Euler (1707 -1783) is a Swiss mathematician.

for any positive integer n . By the same argument, it can be shown that (3) is also true for any nonpositive integer n . Which is known as *de Moivre's³ formula*, and more precisely, we have the following theorem:

Theorem: (de Moivre's Theorem)

For any θ and for any integer n ,

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

In term of exponential form, it essentially reduces to

$$(e^{i\theta})^n = e^{in\theta}.$$

Roots of Complex Numbers

Definition:

Let n be a positive integer ≥ 2 , and let z be nonzero complex number. Then any complex number w that satisfies

$$w^n = z$$

is called the n -th root of z , written as $w = \sqrt[n]{z}$.

Theorem:

Given any nonzero complex number $z = re^{i\theta}$, the equation $w^n = z$ has precisely n solutions given by

$$w_k = \sqrt[n]{r} \left(\cos \left(\frac{\theta + 2k\pi}{n} \right) + i \sin \left(\frac{\theta + 2k\pi}{n} \right) \right), \quad k = 0, 1, \dots, n-1,$$

or

$$w_k = \sqrt[n]{r} \left(e^{i \left(\frac{\theta + 2k\pi}{n} \right)} \right), \quad k = 0, 1, \dots, n-1,$$

where $\sqrt[n]{r}$ denotes the positive real n -th root of $r = |z|$ and $\theta = \text{Arg} z$.

Elementary Complex Functions

Let A and B be sets. A *function* f from A to B , denoted by $f : A \rightarrow B$ is a rule which assigns to *each* element $a \in A$ one and *only one* element $b \in B$, we write

$$b = f(a)$$

and call b the *image* of a under f . The set A is the *domain-set* of f , and the set B is the *codomain* or *target-set* of f . The set of all images

$$f(A) = \{ f(a) : a \in A \}$$

is called the *range* or *image-set* of f . It must be emphasized that both a domain-set and a rule are needed in order for a function to be *well defined*. When the domain-set is not mentioned, we agree that the largest possible set is to be taken.

The Polynomial and Rational Functions

³ This useful formula was discovered by a French mathematician, *Abraham de Moivre* (1667 - 1754).

1. **Complex Polynomial Functions** are defined by

$$P(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n,$$

where $a_0, a_1, \dots, a_n \in \mathbb{C}$ and $n \in \mathbb{N}$. The integer n is called the *degree* of polynomial $P(z)$, provided that $a_n \neq 0$. The polynomial $p(z) = az + b$ is called a *linear function*.

2. **Complex Rational Functions** are defined by the quotient of two polynomial functions; that is,

$$R(z) = \frac{P(z)}{Q(z)},$$

where $P(z)$ and $Q(z)$ are polynomials defined for all $z \in \mathbb{C}$ for which $Q(z) \neq 0$.

In particular, the ratio of two linear functions:

$$f(z) = \frac{az + b}{cz + d} \quad \text{with } ad - bc \neq 0,$$

which is called a *linear fractional function* or *Möbius transformation*.

The Exponential Function

In defining complex exponential function, we seek a function which agrees with the exponential function of calculus when the complex variable $z = x + iy$ is real; that is we must require that

$$f(x + i0) = e^x \quad \text{for all real numbers } x,$$

and which has, by analogy, the following properties:

$$e^{z_1} e^{z_2} = e^{z_1 + z_2},$$

$$e^{z_1} / e^{z_2} = e^{z_1 - z_2}$$

for all complex numbers z_1, z_2 . Further, in the previous section we know that by

Euler's identity, we get $e^{iy} = \cos y + i \sin y$, $y \in \mathbb{R}$. Consequently, combining this we adopt the following definition:

Definition:

Let $z = x + iy$ be complex number. The *complex exponential function* e^z is defined to be the complex number

$$e^z = e^{x+iy} = e^x (\cos y + i \sin y).$$

Immediately from the definition, we have the following properties:

For any complex numbers z_1, z_2 , $z = x + iy$, $x, y \in \mathbb{R}$, we have

$$(i). \quad e^{z_1} e^{z_2} = e^{z_1 + z_2},$$

$$(ii). \quad e^{z_1} / e^{z_2} = e^{z_1 - z_2},$$

$$(iii). \quad |e^{iy}| = 1 \quad \text{for all real } y,$$

$$(iv). \quad |e^z| = e^x,$$

$$(v). \quad \overline{e^z} = e^{\bar{z}},$$

$$(vi). \quad \arg(e^z) = y + 2k\pi, \quad k \in \mathbb{Z},$$

$$(vii). \quad e^z \neq 0,$$

- (viii). $e^z = 1$ if and only if $z = i(2k\pi)$, $k \in \mathbb{Z}$,
 (ix). $e^{z_1} = e^{z_2}$ if and only if $z_1 = z_2 + i(2k\pi)$, $k \in \mathbb{Z}$.

Remark

In calculus, we know that the real exponential function is one-to-one. However e^z is *not* one-to-one on the whole complex plane. In fact, by (ix) it is periodic with period $i(2\pi)$; that is,

$$e^{z+i(2k\pi)} = e^z, \quad k \in \mathbb{Z}.$$

The periodicity of the exponential implies that this function is infinitely many to one.

Trigonometric Functions

From the Euler's identity we know that

$$e^{ix} = \cos x + i \sin x, \quad e^{-ix} = \cos x - i \sin x$$

for every real number x ; and it follows from these equations that

$$e^{ix} + e^{-ix} = 2 \cos x, \quad e^{ix} - e^{-ix} = 2i \sin x.$$

Hence it is natural to define the sine and cosine functions of a complex variable z as follows:

Definition:

Given any complex number z , the *complex trigonometric functions* $\sin z$ and $\cos z$ in terms of complex exponentials are defined to be

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i},$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

Let $z = x + iy$, $x, y \in \mathbb{R}$. Then by simple calculations we obtain

$$\sin z = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} = \sin x \cdot \left(\frac{e^y + e^{-y}}{2} \right) + i \cos x \cdot \left(\frac{e^y - e^{-y}}{2} \right).$$

Hence

$$\sin z = \sin x \cosh y + i \cos x \sinh y.$$

Similarly,

$$\cos z = \cos x \cosh y - i \sin x \sinh y.$$

Also

$$|\sin z|^2 = \sin^2 x + \sinh^2 y, \quad |\cos z|^2 = \cos^2 x + \sinh^2 y.$$

Therefore we obtain

- (i). $\sin z = 0$ if and only if $z = k\pi$, $k \in \mathbb{Z}$;
 (ii). $\cos z = 0$ if and only if $z = (\pi/2) + k\pi$, $k \in \mathbb{Z}$.

The other four trigonometric functions of complex argument are easily defined in terms of sine and cosine functions, by analogy with real argument functions, that is

$$\tan z = \frac{\sin z}{\cos z}, \quad \sec z = \frac{1}{\cos z},$$

where $z \neq (\pi/2) + k\pi$, $k \in \mathbb{Z}$; and

$$\cot z = \frac{\cos z}{\sin z}, \quad \csc z = \frac{1}{\sin z},$$

where $z \neq k\pi$, $k \in \mathbb{Z}$.

As in the case of the exponential function, a large number of the properties of the real trigonometric functions carry over to the complex trigonometric functions. Following is a list of such properties.

For any complex numbers $w, z \in \mathbb{C}$, we have

- (i). $\sin^2 z + \cos^2 z = 1$,
 $1 + \tan^2 z = \sec^2 z$,
 $1 + \cot^2 z = \csc^2 z$;
- (ii). $\sin(w \pm z) = \sin w \cos z \pm \cos w \sin z$,
 $\cos(w \pm z) = \cos w \cos z \mp \sin w \sin z$,
 $\tan(w \pm z) = \frac{\tan w \pm \tan z}{1 \mp \tan w \tan z}$;
- (iii). $\sin(-z) = -\sin z$, $\tan(-z) = -\tan z$,
 $\csc(-z) = -\csc z$, $\cot(-z) = -\cot z$,
 $\cos(-z) = \cos z$, $\sec(-z) = \sec z$;
- (iv). For any $k \in \mathbb{Z}$,
 $\sin(z + 2k\pi) = \sin z$, $\cos(z + 2k\pi) = \cos z$,
 $\sec(z + 2k\pi) = \sec z$, $\csc(z + 2k\pi) = \csc z$,
 $\tan(z + k\pi) = \tan z$, $\cot(z + k\pi) = \cot z$,
- (v). $\overline{\sin z} = \sin \bar{z}$, $\overline{\cos z} = \cos \bar{z}$, $\overline{\tan z} = \tan \bar{z}$,
 $\overline{\sec z} = \sec \bar{z}$, $\overline{\csc z} = \csc \bar{z}$, $\overline{\cot z} = \cot \bar{z}$;

Hyperbolic Functions

The complex hyperbolic functions are defined by a natural extension of their definitions in the real case.

Definition:

For any complex number z , we define the *complex hyperbolic sine* and the *complex hyperbolic cosine* as

$$\sinh z = \frac{e^z - e^{-z}}{2},$$

$$\cosh z = \frac{e^z + e^{-z}}{2}.$$

Let $z = x + iy$, $x, y \in \mathbb{R}$. It is directly from the previous definition, we obtain the following identities:

$$\begin{aligned}\sinh z &= \sinh x \cos y + i \cosh x \sin y, \\ \cosh z &= \cosh x \cos y + i \sinh x \sin y, \\ |\sinh z|^2 &= \sinh^2 x + \sin^2 y, \\ |\cosh z|^2 &= \cosh^2 x + \cos^2 y.\end{aligned}$$

Hence we obtain

(i). $\sinh z = 0$ if and only if $z = i(k\pi)$, $k \in \mathbb{Z}$,

(ii). $\cosh z = 0$ if and only if $z = i\left(\frac{\pi}{2} + k\pi\right)$, $k \in \mathbb{Z}$.

Now, the four remaining complex hyperbolic functions are defined by the equations

$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \operatorname{sech} z = \frac{1}{\cosh z},$$

for $z = i\left(\frac{\pi}{2} + k\pi\right)$, $k \in \mathbb{Z}$;

$$\coth z = \frac{\cosh z}{\sinh z}, \quad \operatorname{csch} z = \frac{1}{\sinh z},$$

for $z = i(k\pi)$, $k \in \mathbb{Z}$.

Immediately from the definition, we have some of the most frequently use identities:

For any complex numbers $w, z \in \mathbb{C}$,

(i). $\cosh^2 z - \sinh^2 z = 1$,

$$1 - \tanh^2 z = \operatorname{sech}^2 z,$$

$$\coth^2 z - 1 = \operatorname{csch}^2 z;$$

(ii). $\sinh(w \pm z) = \sinh w \cosh z \pm \cosh w \sinh z$,

$$\cosh(w \pm z) = \cosh w \cosh z \pm \sinh w \sinh z,$$

$$\tanh(w \pm z) = \frac{\tanh w \pm \tanh z}{1 \pm \tanh w \tanh z};$$

(iii). $\sinh(-z) = -\sinh z$, $\tanh(-z) = -\tanh z$,

$$\operatorname{csch}(-z) = -\operatorname{csch} z, \quad \coth(-z) = -\coth z,$$

$$\cosh(-z) = \cosh z, \quad \operatorname{sech}(-z) = \operatorname{sech} z;$$

(iv). $\overline{\sinh z} = \sinh \bar{z}$, $\overline{\cosh z} = \cosh \bar{z}$, $\overline{\tanh z} = \tanh \bar{z}$,

$$\overline{\operatorname{sech} z} = \operatorname{sech} \bar{z}, \quad \overline{\operatorname{csch} z} = \operatorname{csch} \bar{z}, \quad \overline{\coth z} = \coth \bar{z};$$

Remark

(i). Complex trigonometric and hyperbolic functions are related:

$$\sin iz = i \sinh z, \quad \cos iz = \cosh z, \quad \tan iz = i \tanh z,$$

$$\sinh iz = i \sin z, \quad \cosh iz = \cos z, \quad \tanh iz = i \tan z.$$

(i). The above discussion has emphasized the similarity between the real and their complex extensions. However, this analogy should not be carried too far. For example, the real sine and cosine functions are bounded by 1, i.e.,

$$|\sin x| \leq 1 \quad \text{and} \quad |\cos x| \leq 1 \quad \text{for all } x \in \mathbb{R},$$

but

$$|\sin iy| = |\sinh y| \quad \text{and} \quad |\cos iy| = |\cosh y|$$

which become arbitrary large as $y \rightarrow \infty$.

The Logarithm

One of basic properties of the real-valued exponential function, e^x , $x \in \mathbb{R}$, which is not carried over to the complex-valued exponential function is that of being one-to-one. As a consequence of the periodicity property of complex exponential

$$e^z = e^{z+i(2k\pi)}, \quad z \in \mathbb{C}, \quad k \in \mathbb{Z},$$

this function is, in fact, *infinitely many-to-one*.

Obviously, we cannot define a complex logarithmic as a inverse function of complex exponential since e^z is not one-to-one. What we do instead of define the complex logarithmic not as a *single value ordinary function*, but as a *multivalued relationship* that inverts the complex exponential function; i.e.,

$$w = \log z \quad \text{if} \quad z = e^w,$$

or it will preserve the simple relation

$$e^{\log z} = z \quad \text{for all nonzero } z \in \mathbb{C}.$$

Definition:

Let z be any nonzero complex number. The *complex logarithm* of a complex variable z , denoted $\log z$, is defined to be any of the infinitely many values

$$\log z = \log|z| + i \arg z, \quad z \neq 0.$$

Remark

(i). We can write $\log z$ in the equivalent forms

$$\log z = \log|z| + i(\text{Arg } z + 2k\pi), \quad k \in \mathbb{Z}.$$

(ii). The complex logarithm of zero will remain undefined.

(iii). The logarithm of the real modulus of z is base e (natural) logarithm.

(iv). $\log z$ has infinitely many values consisting of the *unique* real part,

$$\text{Re}(\log z) = \log|z|$$

and the infinitely many imaginary parts

$$\text{Im}(\log z) = \arg z = \text{Arg } z + 2k\pi, \quad k \in \mathbb{Z}.$$

In general, the logarithm of any nonzero complex number is a multivalued relation. However we can restrict the image values so as to defined a *single-value function*.

Definition:

Given any nonzero complex number z , the *principle logarithm function* or the *principle value* of $\log z$, denoted $\text{Log } z$, is defined to be

$$\text{Log } z = \log|z| + i\text{Arg } z,$$

where $-\pi \leq \text{Arg } z < \pi$.

Clearly, by the definition of $\log z$ and $\text{Log } z$, they are related by

$$\log z = \text{Log } z + i(2k\pi), \quad k \in \mathbb{Z}.$$

Let w and z be any two nonzero complex numbers, it is straightforward from the definition that we have the following *identities* of complex logarithm:

(i). $\log(w + z) = \log w + \log z$,

(ii). $\log\left(\frac{w}{z}\right) = \log w - \log z$,

(iii). $e^{\log z} = z$,

(iv). $\log e^z = z + i(2k\pi), \quad k \in \mathbb{Z}$,

(v). $\log(z^n) = n \log z$ for any integer positive n .

Complex Exponents

Definition:

For any fixed complex number c , the *complex exponent* c of a nonzero complex number z is defined to be

$$z^c = e^{c \log z} \quad \text{for all } z \in \mathbb{C} \setminus \{0\}.$$

Observe that we evaluate $e^{c \log z}$ by using the complex exponential function, but since the logarithm of z is multivalued. For this reason, depending on the value of c , z^c may have more than one numerical value.

The *principle value* of complex exponential c , z^c occurs when $\log z$ is replaced by principle logarithm function, $\text{Log } z$ in the previous definition. That is,

$$z^c = e^{c \text{Log } z} \quad \text{for all } z \in \mathbb{C} \setminus \{0\},$$

where $-\pi \leq \text{Arg } z \leq \pi$. If $z = re^{i\theta}$ with $\theta = \text{Arg } z$, then we get

$$z^c = e^{c(\log r + i\theta)} = e^{c \log r} e^{i c \theta}.$$

Inverse Trigonometric and Hyperbolic

In general, complex trigonometric and hyperbolic functions are infinite many-to-one functions. Thus, we define the inverse complex trigonometric and hyperbolic as *multiple-valued relation*.

Definition:

For $z \in \mathbb{C}$, the *inverse trigonometric* $\text{arctrig } z$ or $\text{trig}^{-1} z$ is defined by

$$w = \text{trig}^{-1} z \quad \text{if} \quad z = \text{trig } w.$$

Here, 'trig w ' denotes any of the complex trigonometric functions such as $\sin w$, $\cos w$, etc.

In fact, inverses of trigonometric and hyperbolic functions can be described in terms of logarithms. For instance, to obtain the inverse sine, $\sin^{-1} z$, we write $w = \sin^{-1} z$ when $z = \sin w$. That is, $w = \sin^{-1} z$ when

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i}.$$

Therefore we obtain

$$(e^{iw})^2 - 2iz(e^{iw}) - 1 = 0,$$

that is quadratic in e^{iw} . Hence we find that

$$e^{iw} = iz + (1 - z^2)^{1/2},$$

where $(1 - z^2)^{1/2}$ is a double-valued of z , we arrive at the expression

$$\begin{aligned}\sin^{-1} z &= -i \log\left(iz + (1 - z^2)^{1/2}\right) \\ &= \frac{\pi}{2} \pm i \log\left(z + \sqrt{z^2 - 1}\right).\end{aligned}$$

Here, we have the five remaining inverse trigonometric, as multiple-valued relations which can be expressed in terms of natural logarithms as follows:

$$\begin{aligned}\cos^{-1} z &= \pm i \log\left(z + \sqrt{z^2 - 1}\right), \\ \tan^{-1} z &= \frac{1}{2i} \log\left(\frac{i - z}{i + z}\right), \quad z \neq \pm i, \\ \cot^{-1} z &= \frac{\pi}{2} - \frac{1}{2i} \log\left(\frac{i - z}{i + z}\right), \quad z \neq \pm i, \\ \sec^{-1} z &= \pm i \log\left(\frac{1 + \sqrt{1 - z^2}}{z}\right), \quad z \neq 0, \\ \csc^{-1} z &= \frac{\pi}{2} \pm i \log\left(\frac{1 + \sqrt{1 - z^2}}{z}\right), \quad z \neq 0.\end{aligned}$$

The *principal value of complex trigonometric functions* are defined by

$$\begin{aligned}\operatorname{Arcsin} z &= \frac{\pi}{2} + i \operatorname{Log}\left(z + \sqrt{z^2 - 1}\right), \\ \operatorname{Arcos} z &= i \operatorname{Log}\left(z + \sqrt{z^2 - 1}\right), \\ \operatorname{Arctan} z &= \frac{1}{2i} \operatorname{Log}\left(\frac{i - z}{i + z}\right), \quad z \neq \pm i, \\ \operatorname{Arcot} z &= \frac{\pi}{2} - \frac{1}{2i} \operatorname{Log}\left(\frac{i - z}{i + z}\right), \quad z \neq \pm i, \\ \operatorname{Arcsec} z &= i \operatorname{Log}\left(\frac{1 + \sqrt{1 - z^2}}{z}\right), \quad z \neq 0, \\ \operatorname{Arcsc} z &= \frac{\pi}{2} + i \operatorname{Log}\left(\frac{1 + \sqrt{1 - z^2}}{z}\right), \quad z \neq 0.\end{aligned}$$

Definition:

For any complex number z , the *inverse hyperbolic*, $\operatorname{archyp} z$ or $\operatorname{hyp}^{-1} z$ is defined by

$$w = \operatorname{hyp}^{-1} z \quad \text{if} \quad z = \operatorname{hyp} w.$$

Here ‘hyp’ denotes any of the complex hyperbolic functions such as $\sinh z$, $\cosh z$, etc.

These relations, which are multiple-valued, can be expressed in term of natural logarithms as follows:

$$\begin{aligned}
\sinh^{-1} z &= \begin{cases} \log(z + \sqrt{z^2 + 1}), \\ -\log(z + \sqrt{z^2 + 1}) + i\pi, \end{cases} \\
\cosh^{-1} z &= \pm \log(z + \sqrt{z^2 - 1}), \\
\tanh^{-1} z &= -\frac{1}{2} \log\left(\frac{1-z}{1+z}\right), \quad z \neq \pm 1, \\
\coth^{-1} z &= -\frac{1}{2} \left(i\pi - \log\left(\frac{1+z}{1-z}\right) \right), \quad z \neq \pm 1, \\
\operatorname{sech}^{-1} z &= \pm \log\left(\frac{1 + \sqrt{1 - z^2}}{z}\right), \quad z \neq 0, \\
\operatorname{csch}^{-1} z &= \begin{cases} \log\left(\frac{1 + \sqrt{1 + z^2}}{z}\right), & z \neq 0, \\ -\log\left(\frac{1 + \sqrt{1 + z^2}}{z}\right) + i\pi, & z \neq 0. \end{cases}
\end{aligned}$$

The *principle value* of complex hyperbolic functions are defined by

$$\begin{aligned}
\operatorname{Arcsinh} z &= \operatorname{Log}(z + \sqrt{z^2 + 1}), \\
\operatorname{Arccosh} z &= \operatorname{Log}(z + \sqrt{z^2 - 1}), \\
\operatorname{Arctanh} z &= -\frac{1}{2} \operatorname{Log}\left(\frac{1-z}{1+z}\right), \quad z \neq \pm 1, \\
\operatorname{Arccoth} z &= -\frac{1}{2} \left(i\pi - \operatorname{Log}\left(\frac{1+z}{1-z}\right) \right), \quad z \neq \pm 1, \\
\operatorname{Arcsech} z &= \operatorname{Log}\left(\frac{1 + \sqrt{1 - z^2}}{z}\right), \quad z \neq 0, \\
\operatorname{Arccsch} z &= \operatorname{Log}\left(\frac{1 + \sqrt{1 + z^2}}{z}\right), \quad z \neq 0.
\end{aligned}$$